“Early Universe”: Relics of Preheating after Inflation

Hal Finkel

Argonne National Laboratory

July 11, 2013
1 Introduction
   - Inflation
   - Nonlinear Processes in the Early Universe
   - Preheating

2 Oscillons

3 Oscillons in Monodromy Inflation

4 Gravitational Radiation

5 Conclusion
Outline

1. Introduction
   - Inflation
   - Nonlinear Processes in the Early Universe
   - Preheating

2. Oscillons

3. Oscillons in Monodromy Inflation

4. Gravitational Radiation

5. Conclusion
Cosmology

Afterglow Light Pattern 380,000 yrs.

Inflation

Quantum Fluctuations

Dark Ages

Development of Galaxies, Planets, etc.

1st Stars about 400 million yrs.

Big Bang Expansion

13.7 billion years

Dark Energy Accelerated Expansion

WMAP

(graphic by: NASA/WMAP Science Team)
The distinguishing signatures of individual inflation models are often produced by processes in strongly-nonlinear regimes:

- Density perturbations from preheating...
- and the resulting gravitational radiation.
- and slowly-decaying, localized features such as oscillons.
- Phase transitions and associated processes, such as... bubble nucleation and collisions.
- Primordial black-hole formation
The distinguishing signatures of individual inflation models are often produced by processes in strongly-nonlinear regimes:

- Density perturbations from preheating...
The distinguishing signatures of individual inflation models are often produced by processes in strongly-nonlinear regimes:

- Density perturbations from preheating...
- and the resulting gravitational radiation.
The distinguishing signatures of individual inflation models are often produced by processes in strongly-nonlinear regimes:

- Density perturbations from preheating...
- and the resulting gravitational radiation.
- and slowly-decaying, localized features such as oscillons.
The distinguishing signatures of individual inflation models are often produced by processes in strongly-nonlinear regimes:

- Density perturbations from preheating...
- and the resulting gravitational radiation.
- and slowly-decaying, localized features such as oscillons.
- Phase transitions and associated processes, such as...
The distinguishing signatures of individual inflation models are often produced by processes in strongly-nonlinear regimes:

- Density perturbations from preheating...
- and the resulting gravitational radiation.
- and slowly-decaying, localized features such as oscillons.
- Phase transitions and associated processes, such as...
- bubble nucleation and collisions.
The distinguishing signatures of individual inflation models are often produced by processes in strongly-nonlinear regimes:

- Density perturbations from preheating...
- and the resulting gravitational radiation.
- and slowly-decaying, localized features such as oscillons.
- Phase transitions and associated processes, such as...
- bubble nucleation and collisions.
- Primordial black-hole formation
Energy in the inflaton field needs to be transferred into the fields for normal (standard model) matter and energy (quarks, gluons, electrons, photons, etc.).
Energy in the inflaton field needs to be transferred into the fields for normal (standard model) matter and energy (quarks, gluons, electrons, photons, etc.).

After preheating begins, inflaton (low-\(k\)) occupation numbers are huge, so perturbation theory is ineffective; we need to deal directly with the full nonlinear process.
Energy in the inflaton field needs to be transferred into the fields for normal (standard model) matter and energy (quarks, gluons, electrons, photons, etc.).

After preheating begins, inflaton (low-\(k\)) occupation numbers are huge, so perturbation theory is ineffective; we need to deal directly with the full nonlinear process.

For certain models, this nonperturbative phase is necessary to ensure that reheating completes.
Energy in the inflaton field needs to be transferred into the fields for normal (standard model) matter and energy (quarks, gluons, electrons, photons, etc.).

After preheating begins, inflaton (low-$k$) occupation numbers are huge, so perturbation theory is ineffective; we need to deal directly with the full nonlinear process.

For certain models, this nonperturbative phase is necessary to ensure that reheating completes.

Coupled modes can enter resonance bands which cause resonant amplification.
Model the inflaton-matter system as a set of coupled scalar fields.
Model the inflaton-matter system as a set of coupled scalar fields.
Use classical field theory (as an approximation).
Model the inflaton-matter system as a set of coupled scalar fields. Use classical field theory (as an approximation).

\[ \Box \phi^i = \frac{\partial V}{\partial \phi^i}, \quad V \text{ is a nonlinear function of all of the } \{\phi^i\}. \]
In an FRW background:

\[ ds^2 = -dt^2 + a^2(t) \, d\vec{x}^2 \]  \hspace{1cm} (1)
In an FRW background:

\[ ds^2 = -dt^2 + a^2(t) d\vec{x}^2 \]  \hspace{1cm} (1)

The field equations of motion become:

\[ \ddot{\phi}^i + 3H \dot{\phi}^i - \frac{\Delta}{a^2} \phi^i + \frac{\partial V}{\partial \phi^i} = 0 \]  \hspace{1cm} (2)
$V(\phi, \psi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2$

Using the potential:

$V(\phi, \psi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2$ (3)
\[ V(\phi, \psi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2 \]

Using the potential:

\[ V(\phi, \psi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2 \] (3)

The equation of motion for \( \phi \) is:

\[ \ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + m^2 \phi + g^2 \psi^2 \phi = 0 \] (4)
Using the potential:

\[ V(\phi, \psi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2 \]  

The equation of motion for \( \phi \) is:

\[ \ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + m^2 \phi + g^2 \psi^2 \phi = 0 \]  

In terms of Fourier modes:

\[ \ddot{\phi}_k + 3H \dot{\phi}_k + \left( \frac{k^2}{a^2} + m^2 + g^2 \psi^2 \right) \phi_k = 0 \]
When $H = 0$, substitute:

$$q = \frac{g^2 \psi^2}{4m^2}, \quad A = \frac{k^2}{m^2} + 2q, \quad z = mt$$

(6)
When $H = 0$, substitute:

$$q = \frac{g^2 \psi^2}{4m^2}, \quad A = \frac{k^2}{m^2} + 2q, \quad z = mt$$  \hspace{1cm} (6)

Yielding a Mathieu equation:

$$\phi''_k + (A - 2q \cos(2z))\phi_k = 0$$  \hspace{1cm} (7)

where primes denote differentiation with respect to $z$. 
When $H = 0$, substitute:

$$q = \frac{g^2 \psi^2}{4m^2}, \ A = \frac{k^2}{m^2} + 2q, \ z = mt$$

(6)

Yielding a Mathieu equation:

$$\phi_k'' + (A - 2q \cos(2z))\phi_k = 0$$

(7)

where primes denote differentiation with respect to $z$. All solutions:

$$\phi_k \propto f(z)e^{\pm i \mu z}$$

(8)
$m \geq 0$ implies that $A \geq 2q$. $\phi_k$ grows exponentially if $\mu$ has an imaginary part:

**Figure:** The imaginary part of the Mathieu critical exponent. Outside the heavy black lines the exponent is real-valued. The diagonal line is $A = 2q$. 
Outline

1. Introduction
   - Inflation
   - Nonlinear Processes in the Early Universe
   - Preheating

2. Oscillons

3. Oscillons in Monodromy Inflation

4. Gravitational Radiation

5. Conclusion
A quasi-periodic, localizable feature of a solution to a nonlinear field theory.
Oscillon?

- A quasi-periodic, localizable feature of a solution to a nonlinear field theory.
- Similar to a soliton, but not protected by a symmetry of the Lagrangian.
Oscillon?

- A quasi-periodic, localizable feature of a solution to a nonlinear field theory.
- Similar to a soliton, but not protected by a symmetry of the Lagrangian.
- Mustafa Amin (MIT), has done some of the best recent theory work.
Sextic Oscillon Potential

\[ V(\varphi) = \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi^4 + \frac{g^2}{6m^2}\varphi^6 \]  \hspace{1cm} (9)

with \( \lambda > 0 \) and \( (\lambda/g)^2 \ll 1 \). Assuming spherical symmetry and ignoring expansion gives:

\[ \partial_t^2 \varphi - \partial_r^2 \varphi - \frac{2}{r}\partial_r \varphi + m^2 \varphi - \lambda \varphi^3 + \frac{g^2}{m^2} \varphi^5 = 0 \]  \hspace{1cm} (10)
Sextic-Potential Oscillon Profiles

Assuming a bounded, periodic solution gives an ODE which can be (approximately) solved to yield the radial profile of an oscillon. It is a one-parameter family of curves.

Figure: Oscillon profiles in the sextic potential
Why Do We Care?

**Figure:** The fraction of the energy density of the universe after inflation which is in oscillons. The orange and blue curves are from PSpecRe runs (a lot of them) at $256^3$ and $384^3$ respectively. The black dots are $1024^3$ MPI Defrost runs.
A Universe of Oscillons

Simulation using PSpectRe at $L = 200$ and $N = 256$.

\begin{align*}
a &= 1.85 \\
&\quad \text{a=1.93}
\end{align*}
A Universe of Oscillons (cont.)

\[ a = 2.41 \quad \text{and} \quad a = 3.08 \]
A Universe of Oscillons (cont.)

\[ a = 3.85 \quad \text{and} \quad a = 4.90 \]
Outline

1. Introduction
   - Inflation
   - Nonlinear Processes in the Early Universe
   - Preheating

2. Oscillons

3. Oscillons in Monodromy Inflation

4. Gravitational Radiation

5. Conclusion
V(φ) = m^2 M^2 \left[ \left( 1 + \frac{φ^2}{M^2} \right)^{α} - 1 \right] \quad (11)
Monodromy Inflation

\[ V(\phi) = m^2 M^2 \left[ \left( 1 + \frac{\phi^2}{M^2} \right)^\alpha - 1 \right] \]  

(11)

- Potentials for which \( V(\phi) \sim \phi^{2\alpha} \) with \( \alpha < 1 \) at large \( \phi \) arise in a wide variety of string and supergravity scenarios!
Monodromy Inflation

\[ V(\phi) = m^2 M^2 \left[ \left( 1 + \frac{\phi^2}{M^2} \right)^\alpha - 1 \right] \]  \hspace{1cm} (11)

- Potentials for which \( V(\phi) \sim \phi^{2\alpha} \) with \( \alpha < 1 \) at large \( \phi \) arise in a wide variety of string and supergravity scenarios!

- Quartic inflation \((\alpha = 2)\) is ruled out, and even quadratic inflation \((\alpha = 1)\) is somewhat disfavored, relative to models with \( \alpha < 1 \).
Oscillons can form in potentials of the form:

\[ V(\phi) = \frac{m^2 \phi^2}{2} + U(\phi) \]  \hspace{1cm} (12)

where \( U(\phi) < 0 \) for some interval of the field \( \phi \).
Oscillons can form in potentials of the form:

\[ V(\phi) = \frac{m^2 \phi^2}{2} + U(\phi) \]  \hspace{1cm} (12)

where \( U(\phi) < 0 \) for some interval of the field \( \phi \).

- For our monodromy model this requirement is satisfied if \( \alpha < 1 \).
Oscillons can form in potentials of the form:

\[ V(\phi) = \frac{m^2 \phi^2}{2} + U(\phi) \]  

(12)

where \( U(\phi) < 0 \) for some interval of the field \( \phi \).

- For our monodromy model this requirement is satisfied if \( \alpha < 1 \).
- If \( M \) is significantly sub-Planckian, \( U(\phi) \) is both negative and non-vanishing as the field oscillates about \( \phi = 0 \). This yields resonance and oscillon production!
Figure: The fractional energy density in oscillons after a monodromy-inflation preheating phase as a function of $\alpha$ and $\beta$. 
Figure: Plot from Zhou et al. (2013) - with $\alpha = 1/2$ and $M = 0.01M_P$. The box size is $L = 50/m$ and the energy density isosurface is taken at a value 5 times the average energy density.
Outline

1. Introduction
   - Inflation
   - Nonlinear Processes in the Early Universe
   - Preheating

2. Oscillons

3. Oscillons in Monodromy Inflation

4. Gravitational Radiation

5. Conclusion
Write the metric as:

\[ g_{\mu\nu} = \hat{g}_{\mu\nu} + h_{\mu\nu} \]  \hspace{1cm} (13)

Then metric perturbation obeys:

\[ \Box h_{\alpha\beta} - \hat{g}_{\alpha\beta} \Box h + h_{;\alpha\beta} + 2\hat{R}^{\mu\nu}_{\alpha\beta} h_{\mu\nu} - h_{\alpha\mu;\beta} - h_{\beta\mu;\alpha} + \hat{g}_{\alpha\beta} h_{\mu\nu}^{\mu\nu} = -16\pi G \delta T_{\alpha\beta} \]  \hspace{1cm} (14)

which simplifies after a gauge is chosen.
The stress-energy tensor associated with gravitational radiation is given by:

\[ T_{\mu\nu} = \frac{1}{32\pi G} \langle h_{ij,\mu} h_{ij,\nu} \rangle \]  

(15)

The energy density is given by:

\[ \rho_{gw} = \frac{1}{32\pi G} \langle h_{ij,0} h_{ij,0} \rangle = \sum_{i,j} \frac{1}{32\pi G} \langle h_{ij,0}^2 \rangle \]  

(16)

The fractional contribution to the overall density per logarithmic interval in wave-number:

\[ d\Omega_{gw} \frac{d\ln k}{H^2 L^2} \sum_{i,j} |h_{ij,0}(k)|^2 \]  

(17)
The stress-energy tensor associated with gravitational radiation is given by:

\[ T_{\mu\nu} = \frac{1}{32\pi G} \langle h_{ij,\mu} h_{ij,\nu} \rangle \]  

(15)

The energy density is given by:

\[ \rho_{gw} = \frac{1}{32\pi G} \langle h_{ij,0} h_{ij,0} \rangle = \sum_{i,j} \frac{1}{32\pi G} \langle h_{ij,0}^2 \rangle \]  

(16)
Gravitational-Wave $T_{\mu\nu}$

The stress-energy tensor associated with gravitational radiation is given by:

$$T_{\mu\nu} = \frac{1}{32\pi G} \langle h_{ij,\mu} h_{ij,\nu} \rangle$$ (15)

The energy density is given by:

$$\rho_{gw} = \frac{1}{32\pi G} \langle h_{ij,0} h_{ij,0} \rangle = \sum_{i,j} \frac{1}{32\pi G} \langle h_{ij,0}^2 \rangle$$ (16)

The fractional contribution to the overall density per logarithmic interval in wave-number:

$$\frac{d\Omega_{gw}}{d \ln k} = \frac{1}{\rho_{crit}} \frac{d\rho}{d \ln k} = \frac{\pi k^3}{3H^2 L^2} \sum_{i,j} |h_{ij,0}(k)|^2$$ (17)
Figure: Plot by Price and Siemens showing their results along with results by: Easter, Giblin and Lim; Dufaux, et al.; and García-Bellido, et, al. - a basic $m^2 \phi^2$ model
Figure: Plot by Easther, Giblin and Lim - a basic $m^2 \phi^2$ model - initial energy densities run from $(4.5 \times 10^9 \text{GeV})^4$ to $(4.5 \times 10^{15} \text{GeV})^4$
General Features of Preheating Spectrum

General features of the peak in the gravitational-wave spectrum from preheating:

- GUT-scale inflation: MHz-GHz. Inflation at 10 TeV: $10^{-2}$ Hz. Peak $\approx 1/(\text{inflation scale})$. 

Hal Finkel (ANL)

Early Universe

July 11, 2013
General features of the peak in the gravitational-wave spectrum from preheating:

- GUT-scale inflation: MHz-GHz. Inflation at 10 TeV: $10^{-2}$ Hz. Peak $\approx 1/(\text{inflation scale})$.
- Most power occurs in a narrow frequency band, rapid-drop-off $k^3$ high-frequency tail.
General features of the peak in the gravitational-wave spectrum from preheating:

- GUT-scale inflation: MHz-GHz. Inflation at 10 TeV: $10^{-2}$ Hz. Peak $\approx 1/(\text{inflation scale})$.
- Most power occurs in a narrow frequency band, rapid-drop-off $k^3$ high-frequency tail.
- The higher the inflationary scale the more post-inflation growth takes place and the smaller the wavelength of the resonant modes.
General features of the peak in the gravitational-wave spectrum from preheating:

- GUT-scale inflation: MHz-GHz. Inflation at 10 TeV: $10^{-2}$ Hz. Peak $\approx 1/(\text{inflation scale})$.
- Most power occurs in a narrow frequency band, rapid-drop-off $k^3$ high-frequency tail.
- The higher the inflationary scale the more post-inflation growth takes place and the smaller the wavelength of the resonant modes.
- Maximal production: $\frac{d\Omega_{gw}}{d \ln(k)} \approx 10^{-5}, 10^{-10}$ today.
Figure: Plot by Easther, Giblin and Lim
Figure: Plot from Zhou et al. (2013) - with $\alpha = 1/2$ and $M = 0.01M_P$ at $t = 240/m$. The box size is $L = 25/m$. The vertical lines correspond to the gravitational wave frequencies associated with the different harmonics of the oscillon, which are twice the frequencies of the oscillon harmonics. The values indicated are the frequencies before redshifting to today’s frequencies. - $f_{gw}^{typical} \sim 10^8$ Hz and $h^2\Omega_{gw}^{typical} \sim 10^{-14}$ today.
1 Introduction
   - Inflation
   - Nonlinear Processes in the Early Universe
   - Preheating

2 Oscillons

3 Oscillons in Monodromy Inflation

4 Gravitational Radiation

5 Conclusion
I would like to thank:

- Richard Easther, Mustafa Amin, and my other collaborators.
- Tom Giblin and Eugene Lim, and everyone else who has helped over the years.
- DOE CSGF (who paid for most of this work), and DOE/ANL/ALCF, etc.
“Begin at the beginning and go on till you come to the end: then stop.” - Lewis Carroll, Alice’s Adventures in Wonderland.
Why Inflation?

- Big-Bang cosmology is great, it explains many observations from nucleosynthesis, explains the expanding universe, etc.
Why Inflation?

- Big-Bang cosmology is great, it explains many observations from nucleosynthesis, explains the expanding universe, etc.
- But, it does not explain everything; there are still mysteries:
Why Inflation?

- Big-Bang cosmology is great, it explains many observations from nucleosynthesis, explains the expanding universe, etc.
- But, it does not explain everything; there are still mysteries:
  - The isotropy (homogeneity) of the entire observable universe
Why Inflation?

- Big-Bang cosmology is great, it explains many observations from nucleosynthesis, explains the expanding universe, etc.
- But, it does not explain everything; there are still mysteries:
  - The isotropy (homogeneity) of the entire observable universe
  - The extreme flatness of the observable universe
Why Inflation?

- Big-Bang cosmology is great, it explains many observations from nucleosynthesis, explains the expanding universe, etc.
- But, it does not explain everything; there are still mysteries:
  - The isotropy (homogeneity) of the entire observable universe
  - The extreme flatness of the observable universe
  - The very-nearly-scale-free nature of the initial density-perturbation power spectrum
- $V(\phi)$ has two or more metastable minima with positive vacuum energy.
- $V(\phi)$ has two or more metastable minima with positive vacuum energy.
- Typical regions undergo de Sitter expansion with $H \sim \sqrt{V(\phi_1)/M_p}$ ($M_p$ is the reduced Plank mass, $\phi_1$ is the location of the minimum).
• $V(\phi)$ has two or more metastable minima with positive vacuum energy.

• Typical regions undergo de Sitter expansion with $H \sim \sqrt{V(\phi_1)/M_p}$ ($M_p$ is the reduced Plank mass, $\phi_1$ is the location of the minimum).

• Small regions may tunnel to another minimum $\phi_2$, $V(\phi_2) < V(\phi_1)$, forming a “bubble.”
Figure: Diagram of $V(\phi)$ supporting bubble collisions scenarios. Figure from Easther, et al., 2009
Primordial Black Holes

- Formed after inflation if power spectrum has a small-scale peak.

Produce Hawking radiation as they decay, including gravitational radiation. Could cause a matter-dominated phase.

For average masses larger than \( \approx 1 \text{ gram} \), constrained by nucleosynthesis, x-ray background, dark-matter abundance.
Primordial Black Holes

- Formed after inflation if power spectrum has a small-scale peak.
- Produce Hawking radiation as they decay, including gravitational radiation. Could cause a matter-dominated phase.
Primordial Black Holes

- Formed after inflation if power spectrum has a small-scale peak.
- Produce Hawking radiation as they decay, including gravitational radiation. Could cause a matter-dominated phase.
- For average masses larger than $\approx 1$ gram, constrained by nucleosynthesis, x-ray background, dark-matter abundance.
Treating the full system, including the backreaction from other fields, requires 3-D numerical simulation. We’re neither the first nor the last...
Other Codes

Treating the full system, including the backreaction from other fields, requires 3-D numerical simulation. We’re neither the first nor the last...

- PSpectRe: Easther, Finkel and Roth (2010)
- HLattice: Huang (2011)

- Evolves fields in Fourier space using a second/fourth-order scheme.


- Evolves fields in Fourier space using a second/fourth-order scheme.
- No finite-difference approximations for the derivative terms.

- Evolves fields in Fourier space using a second/fourth-order scheme.
- No finite-difference approximations for the derivative terms.
- Requires lots of FFTs to evaluate the nonlinear terms (uses FFTW or Intel’s MKL).

- Evolves fields in Fourier space using a second/fourth-order scheme.
- No finite-difference approximations for the derivative terms.
- Requires lots of FFTs to evaluate the nonlinear terms (uses FFTW or Intel’s MKL).
- Parallelized using OpenMP.

- Evolves fields in Fourier space using a second/fourth-order scheme.
- No finite-difference approximations for the derivative terms.
- Requires lots of FFTs to evaluate the nonlinear terms (uses FFTW or Intel’s MKL).
- Parallelized using OpenMP.
- Naturally integrates with Fourier-space $h_{\mu\nu}^{TT}$ evolution.
Energy Conservation?

GR’s dynamic metric does not generally allow for a conserved energy. In this case, the FRW background is changing, but homogeneous, and so we have (the averaged Friedmann equation):

\[
\frac{\langle \rho \rangle}{3H^2} - 1
\]

(18)

And it should be as good as the homogeneity assumption (parts in \(10^7\)).
Figure: PSpectRe runs at $32^3$ ($L = 2$ and the time step is 0.005). The red line uses the Verlet integrator, blue shows the Runge-Kutta results.
Figure: PSpectRe: Red is unpadded, blue is padded by a factor of 2.
Figure: Runs with $256^3$ points and $L=10$ (for Defrost’s default model).

PSpectRe’s convergence for equation-of-state observables is better than Defrost’s too (see the paper).
FFT = Fast Fourier Transform (transforms from (discrete) position space to "frequency space")

\[
\Phi(\vec{k}) = \sum_{\vec{r}} \phi(\vec{r}) e^{-i \vec{k} \cdot \vec{r}},
\]

(19)

\[
\phi(\vec{r}) = \frac{1}{N^3} \sum_{\vec{k}} \Phi(\vec{k}) e^{i \vec{k} \cdot \vec{r}}.
\]

(20)

FFT evaluates these using a recursive decomposition: \(O(n \log n)\).

\(\phi\) is real: \(\phi(\vec{r}) = \phi(\vec{r})^*\), so \(\Phi(\vec{k}) = \Phi(-\vec{k})\) and the number of free parameters matches in both representations.
Derivatives in Fourier Space

Each derivative operator brings down a factor of $-ik$, so:

$$\nabla^2 \rightarrow \vec{k} \cdot \vec{k}$$

And so (for example):

$$\int_{\text{box}} |\nabla \phi|^2 = \frac{1}{N^3} \sum_{\vec{k}_{\text{space}}} |\vec{k}|^2 |\Phi(\vec{k})|^2 \quad (21)$$
In the discrete case, there is a complication:

- A discrete (upper-half) mode $k$ corresponds not only to the continuum mode $k$, but also to the continuum mode $k - \frac{2\pi N}{L}$. 
A Complication: Mode Aliasing

In the discrete case, there is a complication:

- A discrete (upper-half) mode $k$ corresponds not only to the continuum mode $k$, but also to the continuum mode $k - \frac{2\pi N}{L}$.

- PSpectRe uses the convention that the first $\frac{N}{2} + 1$ Fourier-space components in any dimension represent the modes $0, \ldots, \frac{\pi N}{L}$ and the remaining $\frac{N}{2} - 1$ points represent the modes $-\frac{\pi(N-2)}{L}$ through $-\frac{2\pi}{L}$. 
A Complication: Mode Aliasing

In the discrete case, there is a complication:

- A discrete (upper-half) mode $k$ corresponds not only to the continuum mode $k$, but also to the continuum mode $k - \frac{2\pi N}{L}$.
- PSpectRe uses the convention that the first $\frac{N}{2} + 1$ Fourier-space components in any dimension represent the modes $0, \ldots, \frac{\pi N}{L}$ and the remaining $\frac{N}{2} - 1$ points represent the modes $-\frac{\pi (N-2)}{L}$ through $-\frac{2\pi}{L}$.
- This works if the modes $\frac{\pi (N+2)}{L}$ through $\frac{2\pi (N-1)}{L}$ are negligible, compared to the modes $-\frac{\pi (N-2)}{L}$ through $-\frac{2\pi}{L}$.
Terms such as $\chi^2 \phi$ are implemented as:

- (Optionally) Pad the Fourier-space grid.
- Perform an inverse FFT (transform to position space).
- Compute the nonlinear operation.
- Perform an FFT (transform to Fourier space).
- (Optionally) Unpad the Fourier-space grid.

Padding in Fourier space is equivalent to performing a polynomial fit using all of the available data points and then filling in using interpolation. It is a bit tricky to implement when using a conjugate-symmetry-reduced storage layout; the details are in the paper.

- Evolves $h^{TT}_{\mu\nu}$ in Fourier space given real-space inputs.

- Evolves $h^{TT}_{\mu\nu}$ in Fourier space given real-space inputs.
- Has stability issues with high-frequency noise.

- Evolves $h_{TT}^{\mu\nu}$ in Fourier space given real-space inputs.
- Has stability issues with high-frequency noise.
- No scalar or vector pieces, no back-reaction.
So what does everyone else do? Generally, they estimate based on calculating $T_{\mu\nu}^{TT}$. There are different techniques:

- Estimate by summing $\rho_{gw}$ produced per $dt$. (Easther and Lim, 2006).
- Estimate by integrating an approximate Green's function for $h^{TT}_{\mu\nu}$ (valid only for modes well inside the horizon). (Dufaux, et al., 2007).
- Estimate by integrating a Green's function for $h^{TT}_{\mu\nu}$ assuming a particular expansion history (matter or radiation dominated, etc.). (Price and Siemens, 2008).
- Assuming Gaussian initial conditions and that (mean) vorticity vanishes, evolve uncoupled $h_{ij}$ (back-reaction included, approx. effect unknown). (García-Bellido, et al., 2008).
What Others Do

So what does everyone else do? Generally, they estimate based on calculating $T_{\mu\nu}$. There are different techniques:

- Estimate by summing $\rho_{gw}$ produced per $dt$. (Easther and Lim, 2006).
- Estimate by integrating approximate Green’s function for $h_{\mu\nu}^{TT}$ (valid only for modes well inside the horizon). (Dufaux, et al., 2007).
So what does everyone else do? Generally, they estimate based on calculating $T_{\mu\nu}^{TT}$. There are different techniques:

- Estimate by summing $\rho_{gw}$ produced per $dt$. (Easther and Lim, 2006).
- Estimate by integrating approximate Green’s function for $h_{\mu\nu}^{TT}$ (valid only for modes well inside the horizon). (Dufaux, et al., 2007).
- Estimate by integrating a Green’s function for $h_{\mu\nu}^{TT}$ assuming a particular expansion history (matter or radiation dominated, etc.). (Price and Siemens, 2008).
So what does everyone else do? Generally, they estimate based on calculating $T_{\mu\nu}$. There are different techniques:

- Estimate by summing $\rho_{gw}$ produced per $dt$. (Easther and Lim, 2006).
- Estimate by integrating approximate Green’s function for $h_{\mu\nu}$ (valid only for modes well inside the horizon). (Dufaux, et al., 2007).
- Estimate by integrating a Green’s function for $h_{\mu\nu}$ assuming a particular expansion history (matter or radiation dominated, etc.). (Price and Siemens, 2008).
- Assuming Gaussian initial conditions and that (mean) vorticity vanishes, evolve uncoupled $h_{ij}$ (back-reaction included, approx. effect unknown). (García-Bellido, et, al., 2008).
The **PSpecRe** run beats the largest **LatticeEasy** run (which had been used for publication).

**Figure:** **PSpecRe**+$h_{ij}$ at $128^3$ (dots) vs. **LatticeEasy**+$h_{ij}$ at $128^3$, $256^3$, $512^3$. The **PSpecRe**+$h_{ij}$ run beats the largest **LatticeEasy**+$h_{ij}$ run (which had been used for publication).
LatticeEASY, Felder and Tkachev (2000).

- The “industry standard.”
LatticeEASY, Felder and Tkachev (2000).

- The “industry standard.”
- Second-order staggered-leapfrog code.
LatticeEASY, Felder and Tkachev (2000).

- The “industry standard.”
- Second-order staggered-leapfrog code.
- Code is functional and well documented.
LatticeEASY, Felder and Tkachev (2000).

- The “industry standard.”
- Second-order staggered-leapfrog code.
- Code is functional and well documented.
- Not performance optimized.

- Second-order Perring-Skyrme-like scheme (plug in a temporal stencil and solve).

- Second-order Perring-Skyrme-like scheme (plug in a temporal stencil and solve).
- Code is clean, although not modular, and well documented.

- Second-order Perring-Skyrme-like scheme (plug in a temporal stencil and solve).
- Code is clean, although not modular, and well documented.
- Careful implementation of initial conditions.

- Second-order Perring-Skyrme-like scheme (plug in a temporal stencil and solve).
- Code is clean, although not modular, and well documented.
- Careful implementation of initial conditions.
- Tuned and optimized for performance.
HLattice, Huang (2011).

- Claims to do scalar fields plus metric perturbations.
HLattice, Huang (2011).

- Claims to do scalar fields plus metric perturbations.
- Uses 6th-order spatial stencil and 4th-order RK integrator.

Preprint posted on Feb. 1, so I'll have more to say after I've tried it...
HLattice, Huang (2011).

- Claims to do scalar fields plus metric perturbations.
- Uses $6^{th}$-order spatial stencil and $4^{th}$-order RK integrator.
- Preprint posted on Feb. 1, so I’ll have more to say after I’ve tried it...
Based on Frolov’s Defrost code, modified by changing array indexing, adding MPI calls, etc.
MPI Defrost

- Based on Frolov’s Defrost code, modified by changing array indexing, adding MPI calls, etc.
- Used for some $1024^3$ oscillon calculations.